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Average Error Bounds of Best Approximation of Continuous Functions on the Wiener Space*

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In this paper, we study the approximation of the identity operator and the integral operator T_m by Jackson operators, discrete Jackson operators, and spline operators, respectively, on the Wiener space and obtain average error estimation.

1. INTRODUCTION

In this paper we study the best approximation of continuous functions by polynomials on the Wiener space in connection with the information-based complexity theory in average case setting developed by Traub, Wasilkowski, and Wozniakowski in [1].

First let us recall some fundamental notions about the information-based complexity in the average case setting. Let F be a set, let G be a normed linear space with norm $\|\cdot\|$, and μ be a probability Borel measure on F. Let S be a measurable mapping from F into G which is called a solution operator. Let N be a measurable mapping from F into $\mathbb{R}^n$ and ϕ be a measurable mapping from $\mathbb{R}^n$ into G which are called an information operator and an algorithm, respectively. For $1 \leq p < +\infty$ the average error of the algorithm ϕ with respect to the measure μ is defined by

$$e_p(S, N, \phi, \mu) := \left(\int_F \|Sx - \phi(N(x))\|_G^p \mu(dx)\right)^{1/p} \tag{1.1}$$

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and the average radius of information N with respect to μ is defined by

$$r_p^{(a)}(N, S, \mu) := \inf_{\phi} e_p(S, N, \phi, \mu), \tag{1.2}$$

where ϕ ranges over the set of all algorithms. If there exists an algorithm $\phi*$ such that

$$e_p(S, N, \phi*, \mu) = r_p^{(a)}(S, N, \mu), \tag{1.3}$$

then we call $\phi*$ an optimal algorithm.

In the last years there has been established a complete theory in the linear case, when F is a separable Banach space, μ is a Gaussian measure on F, and S, N are both bounded linear operators (see Lee, 1986; Lee and Wasilkowski, 1986). Meanwhile, some profound ideas in approximation theory, for example, the notion of width, which was brought forth in approximation theory founded on the concept of uniform error, have been extended to the average case setting and are now intensively studied in connection with the information-based complexity theory (see Mathé, 1990; Majorov, 1992).

Denote

$$C_0[0, 1] := \{f : [0, 1] \to \mathbb{R}, f \text{ continuous on } [0, 1], f(0) = 0\}. \tag{1.4}$$

For every $f \in C_0[0, 1]$ set

$$\|f\|_C := \max_{0 \le t \le 1} |f(t)|.$$

Then $(C_0, \|\cdot\|_C)$ becomes a separable Banach space. Denote by $\mathcal{B}(C_0)$ the Borel class of $(C_0, \|\cdot\|_C)$, and by W the Wiener measure on $\mathcal{B}(C_0)$ (concerning Wiener measure we refer to Kuo, 1975). We shall consider functions taken from C_0 and their integrals. For convenience we introduce integral operator T_m, $m \ge 1$, on C_0 as follows.

Let $m \ge 0$ be an integer. Define

$$(T_0 g)(x) = g(x) \qquad \forall g \in C_0[0, 1]; \tag{1.5}$$

when $m \ge 1$

$$(T_m g)(x) := \frac{1}{(m-1)!} \int_0^1 (x - t)_+^{m-1} g(t)\, dt \qquad \forall g \in C_0[0, 1]. \tag{1.6}$$

Obviously we have

$$(T_m g)(x) \in C_0^m[0, 1] := \{f \in C^m[0, 1]: f^{(j)}(0) = 0, j = 0, \cdots, m\}.$$

To construct Jackson operators on the class $C_0^m[0, 1]$ we take a positive integer $r \geq 2$, another $n > m$ and put $n' = [(n - m)/r] + 1$. Let

$$K_{nr}^{(m)}(t) := \frac{1}{C_{nr}^{(m)}} \left(\sin \frac{n't}{2} \Big/ \sin \frac{t}{2}\right)^{2r}. \tag{1.7}$$

where

$$C_{nr}^{(m)} = \int_{-\pi}^{\pi} \left(\sin \frac{n't}{2} \Big/ \sin \frac{t}{2}\right)^{2r} dt.$$

Now we define a set of Jackson operators $\{J_n^{(m)}\}_{m \geq 0}$ as follows. For $m = 0$ we set

$$J_n^{(0)}(g; x) := J_n(g; x) = \int_{-\pi}^{\pi} g\left(\frac{\cos u + 1}{2}\right) K_{nr}^{(0)}(t - u)\, du; \tag{1.8}$$

when $m \geq 1$,

$$J_n^{(m)}(g; x) = (T_m g)(x)$$
$$- \left(-\frac{1}{2}\right)^m \int_{-\pi}^{\pi} \int_{u_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{u_2}^{u_3} \int_{-\pi}^{\pi} \left[g\left(\frac{\cos u_1 + 1}{2}\right) - g\left(\frac{\cos u_0 + 1}{2}\right)\right]$$
$$\times \prod_{\nu=0}^{m-1} \sin u_{2\nu+1} \prod_{\nu=0}^{m} K_{nr}^{(m)}(u_{2\nu+1} - u_{2\nu}) \cdot \prod_{\nu=0}^{2m} du_\nu, \tag{1.9}$$

where $x = (\cos t + 1)/2$. Denote by $\mathbb{P}_n$ the set of algebraic polynomials of degree $\leq n$. In the next section we will prove that $J_n^{(m)}$ is a linear continuous operator from $C_0[0, 1] \to \mathbb{P}_n$.

Denote the best L_q-approximation

$$E_n(f)_q := \min_{P \in \mathbb{P}_n} \|f - P\|_q, \qquad 1 \leq q \leq \infty$$

(when $q = \infty$ we substitute the sup-norm $\|\cdot\|_C$ for $\|\cdot\|_\infty$).

The L_q-approximation by Jackson operator is given by

$$\|T_m g - J_n^{(m)}(g)\|_q = \begin{cases} (\int_0^1 |T_m g(x) - J_n^{(m)}(g; x)|^q \, dx)^{1/q}, & 1 \le q < \infty, \\ \max_{0 \le x \le 1} |T_m g(x) - J_n^{(m)}(g; x)|, & q = \infty. \end{cases}$$

The average error of the best L_q-approximation of continuous functions by polynomials from $\mathbb{P}_n$ over the Wiener space is given by

$$E_{pq}(T_m, W)_n := \left(\int_{C_0} E_n(T_m g)_q^p w(dg)\right)^{1/p}, \tag{1.10}$$
$$(1 \le p < \infty; \; m \ge 0, \; n > m).$$

The average error of the L_q-approximation by the Jackson operator is given by

$$E_{pq}(T_m, J_n^{(m)}, W)_n := \left(\int_{C_0} \|T_m g - J_n^{(m)}(g)\|_g^p w(dg)\right)^{1/p}, \tag{1.11}$$
$$(1 \le p < \infty; \; m \ge 0, \; n > m).$$

It is obvious that for any $p \in [1, \infty)$ and $q \in [1, +\infty)$ it holds that

$$E_{pq}(T_m, W)_n \le E_{pq}(T_m, J_n^{(m)}, W). \tag{1.12}$$

The fundamental results of this paper may be formulated as follows.

THEOREM 1. *Suppose* $1 \le p, q < +\infty$, $n > m \ge 0$. *Then*

$$E_{pq}(T_m, W)_n \le E_{pq}(T_m, J_n^{(m)}, W) \le Mn^{-m-1/2}. \tag{1.13}$$

For $1 \le p < \infty$, $2 \le q < +\infty$ *we have*

$$E_{pq}(T_m, W)_n \asymp E_{pq}(T_m, J_n^{(m)}, W) \asymp n^{-m-1/2}. \tag{1.14}$$

THEOREM 2. *When* $q = +\infty$, $1 \le p < \infty$, $n > m \ge 0$, *we have*

$$E_{p\infty}(T_m, W)_n \le E_{p\infty}(T_m, J_n^{(m)}, W) \le Mn^{-m-1/2}(\ln n)^{1/2}. \tag{1.15}$$

In the above cases the contant $M > 0$ depends only on p and q, and it takes different values at different places.

The paper is organized into four sections. Section 2 is a preliminary section containing some auxiliary lemmas and useful results. Section 3 contains the proofs of Theorem 1 and 2. In Section 4 we give a discretized version of the Jackson operator $J_n^{(m)}$ and a spline operator defined on $C_0^m[0, 1]$, and get some analogous results for these operators.

2. Preliminaries

As planned in Section 1, in this section we give some auxiliary lemmas and useful results. First we consider Jackson operators defined in (1.8), (1.9). Here, for simplicity of calculation, the Jackson kernel given by (1.7) will be taken as

$$k_{nr}(t) = \frac{1}{C_{nr}} \left(\frac{\sin nt/2}{\sin t/2}\right)^{2r}, \tag{2.1}$$

where $r \geq 2$, $n = 1, 2, 3, \ldots$ are integers, and C_{nr} is a constant such that

$$\int_{-\pi}^{\pi} k_{nr}(t)\, dt = 1. \tag{2.2}$$

Evidently we have

$$k_{nr}(t) = \frac{1}{2\pi} + \sum_{k=1}^{r(n-1)} \rho_{k,r} \cos kt. \tag{2.3}$$

Similar to (1.8) and (1.9) we introduce the operators

$$J_{0,n}(g; x) = \int_{-\pi}^{\pi} g\left(\frac{\cos u + 1}{2}\right) k_{n,r}(t - u)\, du, \tag{2.4}$$

and when $m \geq 1$,

$$\begin{aligned} J_{m,n}(g; x) &= (T_m g)(x) \\ &- \left(\frac{-1}{2}\right)^m \int_{-\pi}^{\pi} \int_{u_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{u_2}^{u_3} \int_{-\pi}^{\pi} \left[g\left(\frac{\cos u_1 + 1}{2}\right) - g\left(\frac{\cos u_0 + 1}{2}\right)\right] \\ &\times \prod_{\nu=0}^{m-1} \sin u_{2\nu+1} \prod_{\nu=0}^{m} k_{nr}(u_{2\nu+1} - u_{2\nu}) \cdot \prod_{\nu=0}^{2m} du_\nu, \end{aligned} \tag{2.5}$$

where $x = (\cos t + 1)/2$. Suppose $n > m$, let $n' = 1 + [(n - m)/r]$. Then

$$J_{m,n'}(g, x) = J_n^{(m)}(g, x). \tag{2.6}$$

LEMMA 1. *For every $g \in C[0, 1]$, $J_{m,n}(g; x)$ is an algebraic polynomial of degree $\le r(n-1) + m$.*

Proof. We prove Lemma 1 by induction. Let $f(u) = g((\cos u + 1)/2)$, then $f(u)$ is an even 2π-periodic continuous function. By (2.3)

$$J_{0,n}(g; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\, du + \sum_{k=1}^{r(n-1)} \rho_{k,r} \cos kt \int_{-\pi}^{\pi} f(u) \cdot \cos ku\, du.$$

So $J_{0,n}(g; x)$ is an algebraic polynomial of degree $\le r(n-1)$. If $m = 1$, then

$$\begin{aligned}
J_{1,n}(g; x) &= (T_1 g)(x) \\
&\quad - \int_{-\pi}^{\pi} \int_{u_2}^{t} g\left(\frac{\cos u_1 + 1}{2}\right)\left(-\frac{1}{2} \sin u_1\right) k_{nr}(t - u_2)\, du_1\, du_2 \\
&\quad - \frac{1}{2} \int_{-\pi}^{\pi} \int_{u_2}^{t} \int_{-\pi}^{\pi} g\left(\frac{\cos u_0 + 1}{2}\right) \\
&\quad \times \sin u_1 k_{nr}(u_1 - u_0)\, k_{nr}(t - u_2)\, du_0\, du_1\, du_2 \\
&= \int_{-\pi}^{\pi} \int_{-\pi}^{u_2} g\left(\frac{\cos u_1 + 1}{2}\right)\left(-\frac{1}{2} \sin u_1\right) k_{nr}(t - u_2)\, du_1\, du_2 \\
&\quad + \int_{-\pi}^{\pi} \int_{u_2}^{t} \int_{-\pi}^{\pi} g\left(\frac{\cos u_0 + 1}{2}\right)\left(-\frac{1}{2} \sin u_1\right) \\
&\quad \times k_{nr}(u_1 - u_0) k_{nr}(u_2 - u_1)\, du_0\, du_1\, du_2 \\
&:= I_1(x) + I_2(x),
\end{aligned}$$

$$\begin{aligned}
&\int_{-\pi}^{-u_2} g\left(\frac{\cos u_1 + 1}{2}\right)\left(-\frac{1}{2} \sin u_1\right) du_1 \\
&= -\int_{u_2}^{\pi} g\left(\frac{\cos u_1 + 1}{2}\right)\left(-\frac{1}{2} \sin u_1\right) du_1 \\
&= \int_{-\pi}^{u_2} g\left(\frac{\cos u_1 + 1}{2}\right)\left(-\frac{1}{2} \sin u_1\right) du_1.
\end{aligned}$$

So from the proof of the case $m = 0$, we know that $I_1(x)$ is an algebraic polynomial of degree $\le r(n-1)$.

$$\begin{aligned}
I_2(x) &= \int_{-\pi}^{\pi} \int_{-\pi}^{t} \int_{-\pi}^{\pi} g\left(\frac{\cos u_0 + 1}{2}\right)\left(-\frac{1}{2} \sin u_1\right) \\
&\quad \times k_{nr}(u_1 - u_0) k_{nr}(t - u_2)\, du_0\, du_1\, du_2
\end{aligned}$$

$$- \int_{-\pi}^{\pi} \int_{-\pi}^{u_2} \int_{-\pi}^{\pi} g\left(\frac{\cos u_0 + 1}{2}\right)\left(-\frac{1}{2}\sin u_1\right)$$
$$\times k_{nr}(u_1 - u_0)k_{nr}(t - u_2)\, du_0\, du_1\, du_2$$
$$= \int_0^x J_{0,n}(g; t)\, dt - \int_{-\pi}^{\pi} \int_{-\pi}^{u_2} \int_{-\pi}^{\pi} g\left(\frac{\cos u_0 + 1}{2}\right)\left(-\frac{1}{2}\sin u_1\right)$$
$$\times k_{nr}(u_1 - u_0)k_{nr}(t - u_2)\, du_0\, du_1\, du_2.$$

Similar to the analysis of $I_1(x)$, we know that $I_2(x)$ is an algebraic polynomial of degree $\leq r(n-1) + 1$. So $J_{1,n}(g; x)$ is an algebraic polynomial of degree $\leq r(n-1) + 1$. Thus Lemma 1 holds for $m = 1$. Suppose that the conclusion of Lemma 1 holds for $m = k$. Then for $m = k + 1$, by (2.5),

$$J_{k+1,n}(g; x) = T(T_k(g; x))$$
$$- \int_{-\pi}^{\pi} \int_{u_2}^{t} \left[T_k\left(g; \frac{\cos u_1 + 1}{2}\right) - J_{k,n}\left(g, \frac{\cos u_1 + 1}{2}\right)\right]$$
$$\times \left(-\frac{1}{2}\sin u_1\right) k_{nr}(t - u_2)\, du_1\, du_2$$
$$= \int_{-\pi}^{\pi} \int_{-\pi}^{t} T_k\left(g; \frac{\cos u_1 + 1}{2}\right)\left(-\frac{1}{2}\sin u_1\right) k_{nr}(t - u_2)\, du_1\, du_2$$
$$- \int_{-\pi}^{\pi} \int_{u_2}^{t} \left[T_k\left(g; \frac{\cos u_1 + 1}{2}\right) - J_{k,n}\left(g; \frac{\cos u_1 + 1}{2}\right)\right]$$
$$\times \left(-\frac{1}{2}\sin u_1\right) k_{nr}(t - u_2)\, du_1\, du_2$$
$$= \int_{-\pi}^{\pi} \int_{-\pi}^{u_2} T_k\left(g; \frac{\cos u_1 + 1}{2}\right)\left(-\frac{1}{2}\sin u_1\right)$$
$$\times k_{nr}(t - u_2)\, du_1\, du_2 + \int_0^x J_{k,n}(g; t)\, dt$$
$$- \int_{-\pi}^{\pi} \int_{-\pi}^{u_2} J_{k,n}\left(g; \frac{\cos u_1 + 1}{2}\right)\left(-\frac{1}{2}\sin u_1\right)$$
$$\times k_{nr}(t - u_2)\, du_1\, du_2$$
$$:= I_1(x) + I_2(x) - I_3(x).$$

Similar to the analysis in the proof of the conclusion of Lemma 1 for the case $m = 1$, we know that $I_1(x)$ and $I_3(x)$ are both algebraic polynomials of degree $\leq r(n-1)$. By the induction hypothesis $I_2(x)$ is an algebraic polynomial of degree $r(n - 1) + k + 1$. So the conclusion of Lemma 1 also holds for $m = k + 1$. So Lemma 1 is proven by induction.

Now substitute n by $n' = [(n - m)/r] + 1$ in $J_{m,n}(g; x)$. By (2.6) and Lemma 1, $J_n^{(m)}(g; x)$ is an algebraic polynomial of degree $\le r(n' - 1) + m = r[(n - m)/r] + m \le n$. We may formulate the following.

PROPOSITION 1. *For any nonnegative integer m and positive integer $n > m$, $J_n^{(m)}$ is a linear continuous operator from $C_0[0, 1] \to \mathbb{P}_n$.*

LEMMA 2. *For k, $1 \le k \le 2r - 2$,*

$$\int_{-\pi}^{\pi} |t|^k K_{n,r}^{(m)}(t)\, dt \asymp n^{-k}, \qquad n \to \infty, \tag{2.7}$$

$$\int_{-\pi}^{\pi} |\cos x - \cos t| K_{n,r}^{(m)}(t)\, dt \le C\frac{1}{n}, \tag{2.8}$$

where $C > 0$ is an absolute constant.

See Sun (1989) and Lorentz (1966).

LEMMA 3. *For every $g \in C[0, 1]$*

$$\|T_m(g) - J_n^{(m)}(g)\|_C \le Mn^{-n}\omega\left(g, \frac{1}{n}\right), \tag{2.9}$$

where $\omega(g, \cdot)$ is the modulous of continuity of g, $M > 0$ is an absolute constant.

Proof. We have

$$\begin{aligned}
&\left| g\left(\frac{\cos u_1 + 1}{2}\right) - g\left(\frac{\cos u_0 + 1}{2}\right)\right| \\
&\quad \le \omega(g, |\cos u_1 - \cos u_0|) \\
&\quad \le [n|\cos u_1 - \cos u_0| + 1]\omega\left(g, \frac{1}{n}\right).
\end{aligned}$$

So by inequality (2.8)

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left| g\left(\frac{\cos u_1 + 1}{2}\right) - g\left(\frac{\cos u_0 + 1}{2}\right)\right| K_{n,r}^{(m)}(u_1 - u_0)\, du_0 \\
&\le \omega\left(g, \frac{1}{n}\right) \int_{-\pi}^{\pi} [n|\cos u_1 - \cos u_0| + 1]\, K_{n,r}^{(m)}(u_1 - u_0)\, du_0 \\
&\le (C + 1)\omega\left(g, \frac{1}{n}\right).
\end{aligned}$$

By inequality (2.7)

$$\begin{aligned}
&|T_m(g; x) - J_n^{(m)}(g; x)| \\
&\leq \frac{1}{2^m}\int_{-\pi}^{\pi}\left|\int_{u_{2m}}^{t}\cdots\int_{-\pi}^{\pi}\left|\int_{u_2}^{u_3}\int_{-\pi}^{\pi}\left|g\left(\frac{\cos u_1 + 1}{2}\right) - g\left(\frac{\cos u_0 + 1}{2}\right)\right|\right.\right. \\
&\quad \times \prod_{\nu=0}^{m} K_{n,r}^{(m)}(u_{2\nu+1} - u_{2\nu})\, du_0\, du_1 \cdots du_{2m-1}\Big|\, du_{2m} \\
&\leq Mn^{-m}\omega\left(g, \frac{1}{n}\right),
\end{aligned}$$

where $x = (\cos t + 1)/2$, $u_{2m+1} = t$.

For $g \in C_0[0, 1]$ and any positive integer n let

$$\begin{aligned}
&L_n(g; x) = g\left(\frac{i-1}{n}\right) + n\left(x - \frac{i-1}{n}\right)\left[g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right], \\
&\frac{i-1}{n} \leq x \leq \frac{i}{n},\ i = 1, 2, \ldots, n.
\end{aligned} \tag{2.10}$$

Then for $(i-1)/n \leq x \leq i/n$.

$$\left|g(x) - g\left(\frac{i-1}{n}\right)\right| \leq |g(x) - L_n(g; x)| + \left|g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right|.$$

LEMMA 4. *For every* $g \in C[0, 1]$

$$\omega\left(g, \frac{1}{n}\right) \leq 2\|g - L_n(g)\|_C + 3\max_{1\leq i\leq n}\left|g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right|. \tag{2.11}$$

The proof of Lemma 4 is very simple. So it is omitted.

LEMMA 5. *For* $1 \leq p < \infty$,

$$\left(\int_{C_0[0,1]}\|g - L_n(g)\|_C^p\, w(dg)\right)^{1/p} = O\left(\frac{\ln n}{n}\right)^{1/2}, \qquad n \to \infty. \tag{2.12}$$

(See K. Ritter 1990, Theorem 2.)

LEMMA 6. *For* $p = 2l$, $l = 1, 2, \ldots$,

$$\int_0^\infty \lambda^{p-1}\left[1 - \left(\sqrt{2/\pi}\int_0^\lambda e^{-t^2/2}\, dt\right)^n\right] d\lambda \leq C_p(\ln n)^{p/2}, \tag{2.13}$$

$$\int_0^\infty \lambda \left[1 - \left(\sqrt{2/\pi} \int_0^\lambda e^{-t^2/2}\,dt\right)^n\right] d\lambda \asymp \ln n, \qquad n \to \infty, \quad (2.14)$$

where $C_p > 0$ is a constant depending only upon p.

Proof. It is easy to verify

$$\frac{\lambda}{1+\lambda^2} e^{-\lambda^2/2} \le \int_\lambda^\infty e^{-t^2/2}\,dt \le \lambda^{-1} e^{-\lambda^2/2}. \quad (2.15)$$

Denote the left side of inequality (2.13) by $I_n(p)$. For $p = 2$, by inequality (2.15)

$$\begin{aligned} I_n(2) &= \int_0^\infty \lambda \sum_{k=0}^{n-1} (\sqrt{2/\pi})^{k+1} \int_\lambda^\infty e^{-t^2/2}\,dt \left(\int_0^\lambda e^{-t^2/2}\,dt\right)^k d\lambda \\ &\le \sum_{k=0}^{n-1} (\sqrt{2/\pi})^{k+1} \int_\lambda^\infty e^{-\lambda^2/2} \left(\int_0^\lambda e^{-t^2/2}\,dt\right)^k d\lambda \\ &= \sum_{k=0}^{n-1} \frac{1}{k+1} = \ln n + O(1). \end{aligned}$$

Now suppose that the inequality (2.13) holds for $p = 2l$. Then for $p = 2(l+1)$,

$$\begin{aligned} I_n(2l+2) &= \int_0^\infty \lambda^{2l+1} \sum_{k=0}^{n-1} (\sqrt{2/\pi})^{k+1} \int_\lambda^\infty e^{-t^2/2}\,dt \left(\int_0^\lambda e^{-t^2/2}\,dt\right)^k d\lambda \\ &\le \sum_{k=0}^{n-1} \int_0^\infty \lambda^{2l} (\sqrt{2/\pi})^{k+1} e^{-\lambda^2/2} \left(\int_0^\lambda e^{-t^2/2}\,dt\right)^k d\lambda \\ &= \sum_{k=0}^{n-1} \frac{\lambda^{2l}}{k+1} \left[\left(\sqrt{2/\pi} \int_0^\lambda e^{-t^2/2}\,dt\right)^{k+1} - 1\right]\Bigg|_{\lambda=0}^\infty \\ &\quad + \sum_{k=0}^{n-1} \frac{2l}{k+1} \int_0^\infty \lambda^{2l-1} \left[1 - \left(\sqrt{2/\pi} \int_0^\lambda e^{-t^2/2}\,dt\right)^{k+1}\right] d\lambda \\ &= \sum_{k=0}^{n-1} \frac{2l}{k+1} I_{k+1}(2l) \le 2lC_{2l} \sum_{k=0}^{n-1} \frac{(\ln(k+1))^l}{k+1} \\ &\le C_{2l+2}(\ln n)^{l+1}, \end{aligned}$$

where $C_{2l+2} = 2lC_{2l}/(l+1)$, So by the induction principle the inequality (2.13) holds for any $p = 2l$, $l = 1, 2, \ldots$.

Also by inequality (2.15),

$$\begin{aligned} I_n(2) &= \sum_{k=0}^{n-1} \int_0^\infty \lambda\,(\sqrt{2/\pi})^{k+1} \int_\lambda^\infty e^{-t^2/2}\,dt \left(\int_0^\lambda e^{-t^2/2}\,dt\right)^k d\lambda \\ &\ge \sum_{k=0}^{n-1} \int_0^\infty \frac{\lambda^2}{1+\lambda^2} e^{-\lambda^2/2} \cdot (\sqrt{2/\pi})^{k+1} \left(\int_0^\lambda e^{-t^2/2}\,dt\right)^k d\lambda \\ &= \sum_{k=0}^{n-1} \frac{1}{k+1}\left[1 - \int_0^\infty \left(\sqrt{2/\pi}\int_0^\lambda e^{-t^2/2}\,dt\right)^{k+1} \frac{2\lambda}{(1+\lambda^2)^2}\,d\lambda\right]. \end{aligned}$$

Since

$$\begin{aligned} &\int_0^\infty \left(\sqrt{2/\pi}\int_0^\lambda e^{-t^2/2}\,dt\right)^{k+1} \frac{2\lambda}{(1+\lambda^2)^2}\,d\lambda \\ &\quad\le \left(\sqrt{2/\pi}\int_0^1 e^{-t^2/2}\,dt\right)^{k+1} \int_0^1 \frac{2\lambda}{(1+\lambda^2)^2}\,d\lambda + \int_1^\infty \frac{2\lambda}{(1+\lambda^2)^2}\,d\lambda \\ &\quad= \frac{1}{2} + \frac{1}{2}\left(\sqrt{2/\pi}\int_0^1 e^{-t^2/2}\,dt\right)^{k+1}; \end{aligned}$$

hence,

$$\begin{aligned} I_n(2) &\ge \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{k+1}\left[1 - \left(\sqrt{2/\pi}\int_0^1 e^{-t^2/2}\,dt\right)^{k+1}\right] \\ &= \frac{1}{2}[\ln n + O(1)]. \end{aligned}$$

This completes the proof of Lemma 6.

LEMMA 7. *For* $p = 2l$, $l = 1, 2, \ldots,$

$$\begin{aligned} &\left(\int_{C_0[0,1]} \max_{1\le i\le n} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^p w(df)\right)^{1/p} \\ &\quad\asymp (\ln n)^{1/2}, \qquad n \to \infty. \end{aligned} \tag{2.16}$$

Proof. Now we consider the distribution function of the random variable $\max_{1\le i\le n} |f(i/n) - f((i-1)/n)|$ with respect to the wiener measure. By the property of independent increment of Brown motion (Kuo, 1975),

$$W\left\{f \in C_0[0, 1], \max_{1\le i\le n} \left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| \ge \lambda\right\}$$
$$= 1 - W\left[\bigcap_{i=1}^{n} \left(\left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| < \lambda\right)\right]$$
$$= 1 - \prod_{i=1}^{n} W \left(\left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| < \lambda\right)$$
$$= 1 - \left(\sqrt{2/\pi}\int_0^{\sqrt{n}\lambda} e^{-t^2/2}\, dt\right)^n.$$

Denote the left side of (2.16) by $I_n(p)$. Put $p = 2l$. Then

$$(I_n(p))^p = \int_{C_0[0,1]} \max_{1\le i\le n} \left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^p w(df)$$
$$= 2l \int_0^\infty \lambda^{2l-1} W \left\{f \in C_0[0, 1]; \max_{1\le i\le n} \left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| \ge \lambda\right\} d\lambda$$
$$= 2l \int_0^\infty \lambda^{2l-1}\left[1 - \left(\sqrt{2/\pi}\int_0^{\lambda} e^{-t^2/2}\, dt\right)^n\right] d\lambda$$
$$= 2l \cdot n^{-l} \int_0^\infty \lambda^{2l-1}\left[1 - \left(\sqrt{2/\pi}\int_0^{\lambda} e^{-t^2/2}\, dt\right)^n\right] d\lambda.$$

By the inequality (2.13), $(I_n(p))^p \le C_{2l}(\ln n/n)^l$, where $C_{2l} > 0$ is a constant only depending upon p $(=2l)$. So

$$\left(\int_{C_0[0,1]} \max_{1\le i\le n} \left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^p w(df)\right)^{1/p}$$
$$= O(\ln n/n)^{1/2}, \qquad n \to \infty,$$

holds for $p = 2l$, $l = 1, 2, \ldots$. On the other hand, by inequality (2.14) $(I_n(2))^2 \ge C_2(\ln n)$. So

$$\left(\int_{C_0[0,1]} \max_{1\le i\le n} \left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^2 w(df)\right)^{1/2} \ge C\left(\frac{\ln n}{n}\right)^{1/2},$$

where $C > 0$ is an absolute constant. By the Hölder inequality, for $p \geq 2$,

$$\left(\int_{C_0[0,1]} \max_{1\leq i\leq n} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^p w(df)\right)^{1/p}$$
$$\geq \left(\int_{C_0[0,1]} \max_{1\leq i\leq n} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^2 w(df)\right)^{1/2}$$
$$\geq C(\ln n/n)^{1/2}.$$

This completes the proof of Lemma 7.

PROPOSITION 2. *Suppose that* $1 \leq p < \infty$; *then*

$$\left(\int_{C_0[0,1]} \omega\left(f, \frac{1}{n}\right)^p w(df)\right)^{1/p} \leq M\left(\frac{\ln n}{n}\right)^{1/2} \tag{2.17}$$

and for $2 \leq p < \infty$,

$$\left(\int_{C_0[0,1]} \omega\left(f, \frac{1}{n}\right)^p w(df)\right)^{1/p} \asymp \left(\frac{\ln n}{n}\right)^{1/2}, \qquad n \to \infty, \tag{2.18}$$

where $M > 0$ *is a constant depending only upon* p.

Proof. By Lemma 7, for $1 \leq p < \infty$, there exists an integer l such that $p \leq 2l$. So by the Hölder inequality,

$$\left(\int_{C_0[0,1]} \max_{1\leq i\leq n} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^p w(df)\right)^{1/p}$$
$$\leq \left(\int_{C_0[0,1]} \max_{1\leq i\leq n} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^{2l} w(df)\right)^{1/2l}$$
$$\leq C(\ln n/n)^{1/2}.$$

By Lemmas 4 and 5, (2.19), and by the Minkowski inequality,

$$\left(\int_{C_0[0,1]} \omega\left(f, \frac{1}{n}\right)^p w(df)\right)^{1/p}$$
$$\leq 2\left(\int_{C_0[0,1]} \|f - L_n(f)\|_C^p w(df)\right)^{1/p}$$

$$+ 3\left(\int_{C_0[0,1]} \max_{1\le i\le n}\left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^p w(df)\right)^{1/p}$$

$$\le C \cdot (\ln n/n)^{1/2}$$

So inequality (2.17) holds. To prove (2.18), because of

$$\max_{1\le i\le n}\left|f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right| \le \omega\left(f, \frac{1}{n}\right),$$

by the Hölder inequality and (2.16), we see that for $2 \le p < \infty$ there exists an absolute contant $C > 0$, such that

$$\left(\int_{C_0[0,1]} \omega\left(f, \frac{1}{n}\right)^p w(df)\right)^{1/p} \ge \left(\int_{C_0[0,1]} \omega\left(f, \frac{1}{n}\right)^2 w(df)\right)^{1/2} \ge C\left(\frac{\ln n}{n}\right)^{1/2}.$$

This completes the proof of Proposition 2.

3. The Proof of Theorems 1 and 2

The Proof of Theorem 1. First we prove the estimation from above in (1.13). Let $p_1 = \max(p, q)$, $1 \le p, q < \infty$; then by the Hölder inequality,

$$E_{p,q}(T_m, J_n^{(m)}, W) \le E_{p_1,p_1}(T_m, J_n^{(m)}, W).$$

Hence from the above inequality, without loss of generality we have only to proof the inequality (1.13) for the case $p = q$. It is known that for the Wiener measure, the following basic property holds: for any $x, y \in [0, 1]$,

$$\int_{C_0[0,1]} f(x)f(y)w(df) = \min(x, y). \tag{3.1}$$

For $p = q$, $x = (\cos t + 1)/2$, we have

$$\begin{aligned}(E_{p,q}(T_m, J_n^{(m)}, W))^p &= \int_{C_0[0,1]}\int_0^1 |T_m f(x) - J_n^{(m)}(f; x)|^p\,dx\,w(df)\\ &= \int_0^1\int_{C_0[0,1]} |T_m f(x) - J_n^{(m)}(f; x)|^p\,w(df)\,dx,\end{aligned}$$

for any fixed $x \in [0, 1]$. Set

$$L(f; x) = T_m f(x) - J_n^{(m)}(f, x).$$

Then $L(f; x)$ is a bounded linear functional on $C_0[0, 1]$ and, since the Wiener measure is a Gaussian measure on $C_0[0, 1]$, as a random variable on the Wiener space, $L(f; x)$ obeys the normal probabilistic distribution $N(0, R(x))$, where

$$\begin{aligned} R(x) &= \int_{C_0[0,1]} [L(f; x)]^2\, w(df) \\ &= \int_{C_0[0,1]} \int_{-\pi}^{\pi} \int_{v_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{v_2}^{v_3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{u_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{u_2}^{u_3} \int_{-\pi}^{\pi} \\ &\quad \times \left[f\left(\frac{\cos u_1 + 1}{2}\right) - f\left(\frac{\cos u_0 + 1}{2}\right)\right] \\ &\quad \times \left[f\left(\frac{\cos v_1 + 1}{2}\right) - f\left(\frac{\cos v_0 + 1}{2}\right)\right] \cdot 2^{-2m} \\ &\quad \times \prod_{j=0}^{m-1} \sin u_{2j+1} \sin v_{2j+1} \prod_{j=0}^{m} K_{n,r}^{(m)}(u_{2j+1} - u_{2j}) K_{n,r}^{(m)}(v_{2j+1} - v_{2j}) \\ &\quad \times \prod_{j=0}^{2m} du_j\, dv_j, \end{aligned}$$

where $u_{2m+1} = v_{2m+1} = t$. From (3.1) and the Fubini theorem, we have

$$\begin{aligned} R(x) &\le 2^{-2m-2} \int_{-\pi}^{\pi} \int_{v_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{v_2}^{v_3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{u_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{u_2}^{u_3} \int_{-\pi}^{\pi} \\ &\quad \times [|\cos u_0 - \cos v_1| - |\cos u_1 - \cos v_1| \\ &\quad + |\cos u_1 - \cos v_0| - |\cos u_0 - \cos v_0|] \\ &\quad \times \prod_{j=0}^{m-1} \sin u_{2j+1} \sin v_{2j+1} \prod_{j=0}^{m} K_{n,r}^{(m)}(u_{2j+1} - u_{2j}) K_{n,r}^{(m)}(v_{2j+1} - v_{2j}) \\ &\quad \times \prod_{j=0}^{2m} du_j\, dv_j \\ &\le 2^{-2m-1} \int_{-\pi}^{\pi} \int_{v_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{v_2}^{v_3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{u_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{u_2}^{u_3} \int_{-\pi}^{\pi} \end{aligned}$$

$$|\cos u_1 - \cos u_0| \prod_{j=0}^{m} K_{n,r}^{(m)}(u_{2j+1} - u_{2j}) K_{n,r}^{(m)}(v_{2j+1} - v_{2j})$$

$$\times \prod_{j=0}^{2m} du_j \, dv_j.$$

By Lemma 2, we have the inequality

$$\int_{-\pi}^{\pi} |t - u| K_{n,r}^{(m)}(t - u)\, du \le C \cdot \frac{1}{n}, \qquad t \in [-\pi, \pi], \tag{3.2}$$

where the constant $C > 0$ is independent of t and n. By (3.2) and (2.8), we have

$$R(x) \le Mn^{-2m-1},$$

where $M > 0$ is a constant depending only upon m. So

$$(E_{p,q}(T_m, J_n^{(m)}, W))^p = \int_0^1 C_p R(x)^{p/2}\, dx \le C_p (Mn^{-2m-1})^{p/2},$$

where $C_p = \pi^{-1/2} 2^{p/2} \Gamma((p + 1)/2)$. Thus we have proved the inequality (1.13).

Now we prove (1.14). From the Hölder inequality, for any $1 \le p \le q < \infty$,

$$E_n(f)_p \le E_n(f)_q \qquad \forall f \in C[0, 1];$$

so for any p, q, $1 \le p < \infty$, $2 \le q < +\infty$,

$$E_{1,2}(T_m, W)_n \le E_{p,q}(T_m, W)_n. \tag{3.3}$$

If there exists some constant $C > 0$ such that

$$E_{1,2}(T_m, W)_n \ge C \cdot n^{-m-1/2}, \tag{3.4}$$

then from (3.3) and (3.4) (1.14) follows. To prove (3.4) we first prove that

$$E_{2,2}(T_m, W)_n \ge C \cdot n^{-m-1/2}, \tag{3.4'}$$

where $C > 0$ is a constant depending only on m.

First we note that, for any $g \in C_0[0, 1]$ such that $\int_0^1 g(t)\, dt = 0$ we have

$$\int_0^1 \int_0^1 \min(s, t) g(s) g(t)\, ds\, dt = \int_0^1 \left(\int_0^x g(t)\, dt\right)^2 dx. \tag{3.5}$$

If $g \perp \mathbb{P}_{m-1}$, then $\forall f \in C[0, 1]$,

$$\int_0^1 g(x) T_m f(x)\, dx = (-1)^m \int_0^1 f(x) T_m g(x)\, dx. \tag{3.6}$$

Now consider the Legendre polynomial $P_n(x)$ on the interval $[0, 1]$ such that $P_n(1) = 1$; then we have the recurrence formula (Sansone, 1959)

$$P'_{n+1}(x) - P'_{n-1}(x) = 2(2n + 1) P_n(x) \tag{3.7}$$

and

$$\int_0^1 [P_n(x)]^2\, dx = \frac{1}{2n + 1}. \tag{3.8}$$

Denote $\hat{X}_n(x) = \sqrt{2n + 1}\, P_n(x)$. Then $\{\hat{X}_n(x)\}_{n=0}^\infty$ forms a normal orthogonal complete system in $L^2[0, 1]$. By (3.7), when $n > m$, $T_m P_n(x) \perp 1$; hence by Parseval equality,

$$\begin{aligned}
&\int_{C_0[0,1]} E_n(T_m f)_2^2 w(df) \\
&= \sum_{k=n+1}^\infty \int_{C_0[0,1]} \left(\int_0^1 (T_m f(x)) \hat{X}_k(x)\, dx\right)^2 w(df) \\
&= \sum_{k=n+1}^\infty \int_{C_0[0,1]} \left(\int_0^1 f(x) (T_m \hat{X}_k(x))\, dx\right)^2 w(df) \\
&= \sum_{k=n+1}^\infty \int_0^1 \int_0^1 \min(s, t) T_m \hat{X}_k(s) T_m \hat{X}_k(t)\, ds\, dt \\
&= \sum_{k=n+1}^\infty \int_0^1 \left(\int_0^x T_m \hat{X}_k(t)\, dt\right)^2 dt \\
&= \sum_{k=n+1}^\infty \int_0^1 (T_{m+1} \hat{X}_k(x))^2\, dx.
\end{aligned} \tag{3.9}$$

Without loss of generality we may assume that $n > 2m + 2$. When $k > 2m + 2$, by the recurrence formula (3.7) we have

$$T_m P_k(x) = \sum_{j=k-m}^{k+m} C_{kj}(m) P_j(x), \tag{3.10}$$

where $C_{kj} = 0$, when m is even, $j + k$ is odd; or when m is odd, $j + k$ is even; and, moreover, it holds that

$$|C_{kj}(m)| \asymp k^{-m}, \qquad k - m \leq j \leq k + m, \text{ when } C_{kj}(m) \neq 0. \tag{3.11}$$

By (3.9)–(3.11)

$$\begin{aligned}(E_{22}(T_m, W)_n)^2 &= \int_{C_0} E_n(T_m f)_2^2 w(df) \\ &= \sum_{k=n+1}^{\infty} (2k+1) \int_0^1 (T_{m+1} P_k(x))^2\, dx \\ &= \sum_{k=n+1}^{\infty} (2k+1) \sum_{j=k-m-1}^{k+m+1} [C_{kj}(m+1)]^2 \cdot \frac{1}{2j+1} \qquad (3.12) \\ &> C(m+1) \sum_{k=n+1}^{\infty} k^{-2m-2} \\ &> C' \cdot n^{-2m-1},\end{aligned}$$

where C and C' are some constants, and C' depends only on m. Thus (3.4′) is proven.

To complete the proof we need the following.

LEMMA 8. *Let μ be a positive finite measure on E and $g \in L(E, \mu)$. If*

(1) $0 < A \leq \int_E g(x)\mu(dx)$.

(2) $0 \leq g(x) \leq M$, $\forall x \in E$.

Then for any $\delta \geq 0$ we have

$$\mu\{x \in E : g(x) \geq \delta A\} \geq \frac{[1 - \delta\mu(E)]A}{M - \delta A}.$$

Proof. Denote

$$\mu\{x \in E : g(x) \geq \lambda\} := m(g; \lambda).$$

Then we have

$$A \le \int_E g(x)\mu(dx) = \int_0^M m(g;\lambda)\,d\lambda = \int_0^{\delta A} m(g;\lambda)\,d\lambda + \int_{\delta A}^M m(g;\lambda)\,d\lambda$$
$$\le \delta A\mu(E) + (M-\delta A)m(g;\delta A).$$

So

$$m(g;\delta A) \ge [1-\delta\mu(E)]A/(M-\delta A). \tag{3.13}$$

Now we prove (3.4). Because of

$$\int_0^x T_m\hat{X}_k(t)\,dt = \int_0^1 \frac{1}{m!}(x-t)_+^m\hat{X}_k(t)\,dt,$$

denote

$$g_n(x) = \sum_{k=n+1}^{\infty}\left(\int_0^1 \frac{1}{m!}(x-t)_+^m\hat{X}_k(t)\,dt\right)^2.$$

We have from (3.9) and (3.11)

$$\int_{C_0} E_n(T_m f)_2^2 w(df) = \int_0^1 g_n(t)\,dt \ge C'n^{-2m-1}.$$

From the recurrence formula

$$P'_{n+1}(x) - P'_{n-1}(x) = 2(2n+1)P_n(x) \tag{3.14}$$

by simple calculation we get

$$\begin{aligned}\int_0^1 \frac{1}{m!}&(x-t)_+^m\hat{X}_k(t)\,dt\\ &= \sum_{j=k-m}^{k+m} C_{kj}(m)\cdot\frac{1}{2(2j+1)}[P_{j+}(x) - P_{j-1}(x)]\cdot\sqrt{2k+1}.\end{aligned}\tag{3.15}$$

Then making use of

$$|P_n(x) - P_{n+2}(x)| \le \frac{4}{\sqrt{(n+2)\pi}} \tag{3.16}$$

we have

$$\left|\int_0^1 \frac{1}{m!}(x-t)_+^m \hat{X}_k(t)\,dt\right| \le C'' \cdot k^{-m-1}. \tag{3.17}$$

Thus from (3.11), (3.15)–(3.17) we obtain

$$g_n(x) \le B \cdot n^{-2m-1}. \tag{3.18}$$

By Lemma 8

$$\left|\left\{x \in [0,1] : g_n(x) \ge \frac{C'}{2} \cdot n^{-2m-1}\right\}\right| \ge \frac{1}{2} C' \cdot n^{-2m-1} \Big/ \left(B - \frac{1}{2} C'\right) n^{-2m-1}$$
$$= \frac{C'}{2B - C'} = C > 0.$$

It follows that

$$\int_{C_0} E_n(T_m)_2 w(df) = \int_0^1 \sqrt{g_n(x)}\,dx \ge C \cdot n^{-2m-1/2}.$$

for some constant $C > 0$; (3.4) is proven.

The Proof of Theorem 2. By Lemma 3, there exists a constant $C > 0$ such that

$$\|T_m f - J_n^{(m)}(f)\|_c \le C\omega(f, 1/n), \tag{3.12}$$

and by Proposition 2, for $1 \le p < \infty$,

$$E_{p,\infty}(T_m, J_n^{(m)}, W) = \left(\int_{C_0[0,1]} \|T_m f - J_n^{(m)} f\|_C^p w(df)\right)^{1/p}$$
$$\le C \left(\int_{C_0[0,1]} \left[\omega\left(f, \frac{1}{n}\right) \cdot n^{-m}\right]^p w(df)\right)^{1/p}$$
$$\le C \cdot M n^{-m} (\ln n/n)^{1/2},$$

where $M > 0$ is the constant given in Proposition 2. Thus we have proved Theorem 2.

4. Spline Algorithm and Discrete Jackson Operators

In this section we are concerned about the connection of the above obtained results with the information-based complexity theory in the average case setting. We first introduce a collection of informations defined on $C_0^m[0, 1]$ as follows. Let $n > m \geq 0$ be some integers, $n^* = [n/(m + 1)]$. For a set of points $0 < x_1 < \cdots < x_{n^*} \leq 1$ define

$$N_k(f) := [(T_k f)(x_j): 0 \leq k \leq m,\ 1 \leq j \leq n^*], \tag{4.1}$$

where T_k is defined as in (1.5) and (1.6), i.e., for $f \in C_0[0, 1]$,

$$(T_0 f)(x_j) = f(x_j), \qquad 1 \leq j \leq n^*,$$

if $k \geq 1$,

$$(T_k f)(x_j) = \frac{1}{(k-1)!} \int_0^1 (x_j - t)_+^{k-1} f(t)\, dt.$$

It is obvious that every $(T_k f)(x_j)$ is a bounded linear functional on $C_0[0, 1]$, so that $N_n(f)$ is a linear information operator from $C_0[0, 1]$ into $\mathbb{R}^n$, whose cardinal $(m + 1)n^* \leq n$. The collection of all such N_n is denoted by N_m^{st}. Let ϕ be a measurable mapping from $\mathbb{R}^{(m+1)n^*}$ into $L_q[0, 1]$, $1 \leq q \leq \infty$. For a given $N_n \in N_m^{st}$, the operator $\phi(N_n(f))$ from $C_0[0, 1] \to L_q[0, 1]$ is said to be an algorithm based on $N_n(f)$. Then the average error of algorithm ϕ over the Wiener space is given by

$$E_{p,q}^{(a)}(T_m, N_n, \phi, W) = \left(\int_{C_0[0,1]} \|T_m f - \phi(N_n(f))\|_q^p w(df) \right)^{1/p} \tag{4.2}$$

The average radius of information of N_n on the Wiener space is

$$r_{p,q}^{(a)}(T_m, N_n, W) = \inf_{\phi} E_{p,q}^{(a)}(T_m, N_n, \phi, W), \tag{4.3}$$

where ϕ ranges over all measurable mappings from $\mathbb{R}^{(m+1)n^*}$ into $L_q[0, 1]$. The optimal n-radius of standard information is given by

$$r_{pq}^{st}(T_m, W)_n = \inf_{N_n \in N_m^{st}} r_{pq}^{(a)}(T_m, N_n, W). \tag{4.4}$$

We put

$$K_n(x, t) = -(-t)_+^0 + n^* \sum_{k=1}^{n^*} \left[\left(x - \frac{k-1}{n^*}\right)_+ - \left(x - \frac{k}{n^*}\right)_+\right]$$
$$\times \left[\left(x - \frac{k}{n^*}\right)_+^0 - \left(x - \frac{k-1}{n^*}\right)_+^0\right], \qquad 0 \le x, t \le 1. \tag{4.5}$$

Define a one-degree spline interpolating operator $\forall f \in C_0[0, 1]$:

$$S_n(f; x) := \int_0^1 f(t) K_n(x, dt). \tag{4.6}$$

Then $S_n(f, x)$ can be represented as

$$S_n(f, x) = f\left(\frac{k-1}{n^*}\right) + n^* \left(x - \frac{k-1}{n^*}\right) \left[f\left(\frac{k}{n^*}\right) - f\left(\frac{k-1}{n^*}\right)\right],$$
$$\frac{k-1}{n^*} \le x \le \frac{k}{n^*}, \qquad k = 1, 2, \ldots, n^*. \tag{4.7}$$

So for every $f \in [0, 1]$,

$$S_n\left(f, \frac{k}{n^*}\right) = f\left(\frac{k}{n^*}\right), \qquad k = 1, 2, \ldots, n^*, \tag{4.8}$$

and S_n is a positive bounded linear operator from $C[0, 1]$ into $C[0, 1]$. And, moreover,

$$S_n(1; x) = 1, \qquad 0 \le x \le 1. \tag{4.9}$$

Set $F_x(t) = |x - t|$, $x, t \in [0, 1]$; then

$$0 \le S_n(F_x(\cdot), x) \le (2n^*)^{-1}, \tag{4.10}$$

and for every $f \in C[0, 1]$,

$$\|f - S_n(f)\|_C \le \omega(f, 1/n^*). \tag{4.11}$$

In fact, suppose that $(k-1)/n^* \le x \le k/n^*$, $1 \le k \le n^*$, by (4.7):

$$S_n(F_x(\cdot),x) = 2n^* \left(\frac{k}{n^*} - x\right)\left(x - \frac{k-1}{n^*}\right) \le (2n^*)^{-1}$$

and

$$|f(x) - S_n(f; x)| \le n^* \left(\frac{k}{n^*} - x\right) \left|f(x) - f\left(\frac{k-1}{n^*}\right)\right| + n^* \left(x - \frac{k-1}{n^*}\right) \left|f(x) - f\left(\frac{k}{n^*}\right)\right| \le \omega\left(f, \frac{1}{n^*}\right).$$

So (4.10 and (4.11) hold.

Based on (4.6), we introduce an operator $S_n^{(m)}$ as follows: For $m \ge 1$,

$$T_m f - S_n^{(m)}(f) = (I - S_n)T(I - S_n)T \cdots (I - S_n)T(I - S_n)f, \tag{4.12}$$

where in (4.12) I is identity, the product in the right side has $(2m + 1)$ factors, in which T is taken m times, and $I - S_n$ $(m + 1)$ times. Then we can represent (4.12) as

$$\begin{aligned} T_m f(x) &- S_n^{(m)}(f; x) \\ &= \int_0^1 \int_{u_{2m}}^x \cdots \int_0^1 \int_{u_2}^{u_3} \int_0^1 [f(u_1) - f(u_0)] K_n(u_1, du_0)\, du_1 \\ &\times K_n(u_3, du_2) \cdots du_{2m-1} K_n(u_{2m+1}, du_{2m}), \end{aligned} \tag{4.13}$$

where $u_{2m+1} = x$. Because for any $x_0 \in [0, 1]$,

$$\int_0^x (t - x_0)_+^k \, dt = \frac{(x - x_0)_+^{k+1}}{k + 1}, \qquad k = 0, 1, 2, \ldots,$$

expanding the product in (4.12), we see that $S_n^{(m)}(f, x)$ is an $(m + 1)$-degree polynomial spline with knots $\{0, 1/n^*, 2/n^*, \ldots, 1\}$. For any $f \in C_0[0, 1]\, T_m f(x) \in C_0[0, 1]$, so that $S_n^{(m)}(f, x)$ is a polynomial spline algorithm based on standard information,

$$N_n^*(f) = \left[T_k f\left(\frac{j}{n^*}\right) : 0 \le k \le m,\ 1 \le j \le n^*\right]$$

and satisfies the interpolation conditions

$$S_n^{(m)}\left(f, \frac{k}{n^*}\right) = T_m f\left(\frac{k}{n^*}\right), \qquad k = 1, 2, \ldots, n^*.$$

Proposition 3. *For all $f \in C[0, 1]$*

$$\|T_m f - S_n^{(m)}(f)\|_C \leq (2n^*)^{-m}\omega\left(f, \frac{1}{n^*}\right). \tag{4.14}$$

Proof. By (4.11), for any $u_1 \in [0, 1]$,

$$\int_0^1 |f(u_1) - f(u_0)| K_n(u_1, du_0) \leq \omega\left(f, \frac{1}{n^*}\right).$$

By (4.7) and (4.10), $\forall x \in [0, 1]$,

$$|T_m f(x) - S_n(f, x)| \leq (2n^*)^{-m}\omega\left(f, \frac{1}{n^*}\right).$$

So inequality (4.14) holds.

Considering the average error of T_m approximated by $S_n^{(m)}$ over the Wiener space, we have the following.

Theorem 3. *For $1 \leq p < \infty$,*

$$\begin{aligned} r_{p\infty}^{st}(T_m, W)_n &\leq \left(\int_{C_0[0,1]} \|T_m f - S_n^{(m)} f\|_C^p w(df)\right)^{1/p} \\ &= O(n^{-m}(\ln n/n)^{1/2}), \qquad n \to \infty. \end{aligned} \tag{4.15}$$

Proof. By Proposition 3, there exists an absolute constant $C > 0$ such that for any $f \in C[0, 1]$,

$$\|T_n f - S_n^{(m)} f\|_C \leq Cw(f, 1/n).$$

So by Proposition 2,

$$\begin{aligned} \left(\int_{C_0[0,1]} \|T_m f - S_n^{(m)} f\|_C^p w(df)\right)^{1/p} &\leq Cn^{-m}\left(\int_{C_0[0,1]} \omega\left(f, \frac{1}{n}\right)^p w(df)\right)^{1/p} \\ &= O(n^{-m}(\ln n/n)^{1/2}), \qquad n \to \infty. \end{aligned}$$

This finishes the proof of Theorem 3.

THEOREM 4. *For* $1 \le p, q < \infty$, $r = \max(p, q)$, *we have*

$$\begin{aligned} r_{pq}^{s,t}(T_m, W)_n &\le \left(\int_{C_0[0,1]} \|T_m f - S_n^{(m)} f\|_q^p w(df)\right)^{1/p} \\ &\le C_r (2n^*)^{-m-1/2}, \end{aligned} \tag{4.16}$$

where in (4.16), $n^* = [n/(m+1)]$, $(C_r)^r = \pi^{-1/2} 2^{r/2} \Gamma((r+1)/2)$.

Proof. Denote

$$E_{p,q}(T_m, S_n^{(m)}, W) = \left(\int_{C_0[0,1]} \|T_m f - S_n^{(m)} f\|_q^p w(df)\right)^{1/p}.$$

By the Hölder inequality, for $r = \max(p, q)$,

$$E_{p,q}(T_m, S_n^{(m)}, W) \le E_{r,r}(T_m, S_n^{(m)}, W)$$

and

$$\begin{aligned} (E_{r,r}(T_m, S_n^{(m)}, W))^r &= \int_{C_0[0,1]} \int_0^1 |T_m f(x) - S_n^{(m)}(f, x)|^r \, dx \, w(df) \\ &= \int_0^1 \int_{C_0[0,1]} |T_m f(x) - S_n^{(m)}(f, x)|^r w(df) \, dx. \end{aligned}$$

Since the Wiener measure is Gaussian, so for any fixed $x \in [0, 1]$, the bounded linear functional $T_m f(x) - S_n^{(m)}(f, x)$ as a random variable on the Wiener space obeys the normal distribution $N(0, R_n^{(m)}(x))$, where

$$\begin{aligned} R_n^{(m)}(x) &= \int_{C_0[0,1]} |T_m f(x) - S_n^{(m)}(f, x)|^2 w(df) \\ &= \int_{C_0[0,1]} \int_0^1 \int_{u_{2m}}^x \cdots \int_0^1 \int_{u_2}^{u_1} \int_0^1 \int_3^1 \int_{v_{2m}}^x \cdots \int_0^1 \int_{v_2}^{v_1} \int_0^1 \\ &\quad \times [f(u_1) - f(u_0)][f(v_1) - f(v_0)] K_n(u_1, du_0) K_n(v_1, dv_0) \, du_1 \, dv_1 \\ &\quad \times K_n(u_3, du_2) K_n(v_3, dv_2) \cdots du_{2m-1} \, dv_{2m-1} K_n(x \, du_{2m}) \\ &\quad \times K_n(x, dv_{2m}) w(df). \end{aligned}$$

By (3.1) and (4.10) and the inequality

$$\big| |u_1 - v_0| + |u_0 - v_1| - |u_1 - v_1| - |u_0 - v_0| \big| \le 2|u_0 - v_0|,$$

we have

$$R_n^{(m)}(x) \le \frac{1}{2}\int_0^1\int_{u_{2m}}^x \cdots \int_0^1\int_{u_2}^{u_1}\int_0^1 \cdot \int_0^1\int_{v_{2m}}^x \cdots \int_0^1\int_{v_2}^{v_3}\int_0^1 [|u_1 - v_0| + |u_0 - v_1| - |u_1 - v_1| - |u_0 - v_0|] \times K_n(u_1, du_0)K_n(v_1, dv_0)\, du_1\, dv_1 \times K_n(u_3, du_2)K_n(v_3, dv_2) \cdots du_{2m-1}\, dv_{2m-1} \times K_n(x, du_{2m})K_n(x, dv_{2m}) \le (2n^*)^{-2m-1} \qquad \forall x \in [0, 1].$$

Thus

$$(E_{r,r}(T_m, S_n^{(m)}, W))^r = \int_0^1 (C_r)^r(R_n^{(m)}(x))^{r/2}\, dx \le (C_r \cdot (2n^*)^{-m-1/2})^r.$$

Hence

$$E_{p,q}(T_m \cdot S_n^{(m)}, W) \le E_{r,r}(T_m, S_n^{(m)}, W) \le C_r(2n^*)^{-m-1/2}.$$

The proof of Theorem 4 is completed.

Assume that $k < N$ are two positive integers; then

$$\sum_{j=1}^{2N} \cos k \cdot \frac{j\pi}{N} = \sum_{j=1}^{2N} \sin k \cdot \frac{j\pi}{N} = 0. \tag{4.17}$$

Let $r \ge 2$ be an integer, taking $n' = [(n - m)/r] + 1$, the Jackson kernel given in (1.7) can be represented as

$$K_{n,r}^{(m)}(t) = \frac{\pi}{C_{nr}}\left(\frac{\sin n't/2}{\sin t/2}\right)^{2r} = \frac{1}{\pi}\left\{\frac{1}{2} + \sum_{k=1}^{n-m} \rho_{kr}\cos kt\right\}. \tag{4.18}$$

Define

$$\begin{aligned} K_{n,r}(x, t) &= \frac{\pi}{n}\sum_{j=-n}^{n}{}'' K_{nr}^{(m)}\left(x - \frac{j\pi}{n}\right)\left(t - \frac{j\pi}{n}\right)_+^0 \\ &= \frac{\pi}{nC_{nr}}\sum_{j=-n}^{n}{}'' \left(\frac{\sin(n'/2)(x - j\pi/n)}{\sin(1/2)(x - j\pi/n)}\right)^{2r}\left(t - \frac{j\pi}{n}\right)_+^0, \end{aligned} \tag{4.19}$$

where the symbol Σ'' means that in the sum the coefficients of the first and the final term should be multiplied by $\frac{1}{2}$.

By (4.17) and (4.18)

$$\sum_{j=-n}^{n}{}'' K_{nr}\left(x - \frac{j\pi}{n}\right) = 1,$$

which is equivalent to

$$\int_{-\pi}^{\pi} K_{nr}(x, dt) = 1, \qquad x \in \mathbb{R}. \tag{4.20}$$

There exists an absolute constant $M > 0$ such that for any $x \in [-\pi, \pi]$,

$$\int_{-\pi}^{\pi} |x - t| K_{n,r}(x, dt) \leq M \cdot \frac{1}{n}, \tag{4.21}$$

and for every $x \in \mathbb{R}$,

$$\int_{-\pi}^{\pi} |\cos x - \cos t| K_{nr}(x, dt) \leq M \cdot \frac{1}{n}. \tag{4.22}$$

In fact, put $S_1 = \{j \in \mathbb{Z} : |j| \leq n, |x - j\pi/n| \leq \pi/n\}$, $S_2 = \{j \in \mathbb{Z} : |j| \leq n, |x - j\pi/n| > \pi/n\}$, where $\mathbb{Z}$ is the set of all integers, then S_1 at most contains two elements. Since $C_{nr} \asymp n^{2r-1}$,

$$\begin{aligned}
\int_{-\pi}^{\pi} |x - t| K_{nr}(x, dt) &= \frac{\pi}{n} \sum_{j=-n}^{n}{}'' K_{nr}^{(m)}\left(x - \frac{j\pi}{n}\right) \left|x - \frac{j\pi}{n}\right| \\
&\leq \sum_{j \in S_1} \frac{\pi}{n} K_{nr}^{(m)}\left(x - \frac{j\pi}{n}\right) \left|x - \frac{j\pi}{n}\right| \\
&\quad + \frac{\pi}{n} \sum_{j \in S_2} K_{nr}^{(m)}\left(x - \frac{j\pi}{n}\right) \left|x - \frac{j\pi}{n}\right| \\
&:= I_n(1) + I_n(2).
\end{aligned}$$

Since

$$\left(\sin \frac{n'}{2} t / \sin \frac{t}{2}\right)^{2r} \leq (n')^{2r},$$

obviously

$$I_n(1) = \sum_{j\in S_1} \frac{\pi}{nC_{nr}} \left(\frac{\sin(n'/2)(x - j\pi/n)}{\sin(1/2)(x - j\pi/n)}\right)^{2r} \le \frac{\pi}{nC_{nr}} \cdot \frac{\pi}{n} \cdot (n')^{2r} \le C \cdot \frac{1}{n},$$

where $C > 0$ is an absolute constant;

$$I_n(2) = \frac{\pi}{nC_{nr}} \sum_{j\in S_2} \left|x - \frac{j\pi}{n}\right| \left(\frac{\sin(n'/2)(x - j\pi/n)}{\sin(1/2)(x - j\pi/n)}\right)^{2r}$$

$$\le \frac{\pi}{nC_{nr}} \sum_{j\in S_2} \frac{\pi^{2\nu}}{|x - j\pi/n|^{2r-1}} \le \frac{C'}{n^{2r}} \sum_{k=1}^{\infty} \frac{n^{2r-1}}{k^{2r-1}} \le C''/n,$$

where C'' is also an absolute constant. So (4.21) holds; (4.22) is an immediate consequence of (4.21).

By (4.21) and (4.22), it is easy to prove that, for every 2π-periodic continuous function $f(x)$,

$$\max_x \left|f(x) - \int_{-\pi}^{\pi} f(t)K_{nr}(x, dt)\right| \le M\omega\left(f, \frac{1}{n}\right), \tag{4.23}$$

where $M > 0$ is a constant independent of f and n.

Let $x = (\cos t + 1)/2$, for any $f \in C[0, 1]$, we introduce discrete Jackson-type operators as follows. For $m = 0$,

$$J_{n,0}^{(d)}(f, x) := J_n^{(d)}(f, x) = \int_{-\pi}^{\pi} f\left(\frac{\cos u_0 + 1}{2}\right) K_{nr}(t, du_0); \tag{4.24}$$

for $m \ge 1$,

$$\begin{aligned} &J_{n,m}^{(d)}(f, x) \\ &= T_m f(x) - \left(-\frac{1}{2}\right)^m \int_{-\pi}^{\pi} \int_{u_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{u_2}^{u_3} \int_{-\pi}^{\pi} \\ &\quad \times \left[f\left(\frac{\cos u_1 + 1}{2}\right) - f\left(\frac{\cos u_0 + 1}{2}\right)\right] \\ &\quad \times \prod_{j=0}^{m-1} \sin u_{2j+1} K_{nr}(u_1, du_0)\, du_1 K_{nr}(u_3, du_2) \cdots du_{2m-1} K_{nr}(t, du_{2m}). \end{aligned} \tag{4.25}$$

Similar to the proof of Lemma 1 and Proposition 1, it is easy to prove that $J_{n,m}^{(d)}$ is a bounded linear operator from $C[0, 1]$ into $\mathbb{P}_n$. By (4.23), for $\forall u_1 \in \mathbb{R}$,

$$\int_{-\pi}^{\pi} \left| f\left(\frac{\cos u_1 + 1}{2}\right) - f\left(\frac{\cos u_0 + 1}{2}\right) \right| K_{nr}(u_1, du_0) \leq M \cdot \frac{1}{n},$$

and by (4.22), there exists an absolute constant $C > 0$ such that, for every $f \in C[0, 1]$,

$$\|T_m f - J_{nm}^{(d)}(f)\|_C \leq C n^{-m} \omega(f, 1/n). \tag{4.26}$$

For the average error of T_m approximated by $J_{n,m}^{(d)}$ on the Wiener space, we have the following results.

THEOREM 5. *For* $1 \leq p < \infty$,

$$\left(\int_{C_0[0,1]} \|T_m f - J_{n,m}^{(d)}(f)\|_C^p w(df) \right)^{1/p} = O\left(n^{-m} \left(\frac{\ln n}{n} \right)^{1/2} \right), \quad n \to \infty. \tag{4.27}$$

Proof. By (4.26) and Proposition 2,

$$\begin{aligned} \left(\int_{C_0[0,1]} \|T_m f - J_{n,m}^{(d)}(f)\|_C^p w(df) \right)^{1/p} &\leq C n^{-m} \left(\int_{C_0[0,1]} w\left(f, \frac{1}{n}\right)^p \omega(df) \right)^{1/p} \\ &= O(n^{-m}(\ln n/n))^{1/2}, \qquad (n \to \infty). \end{aligned}$$

Thus Theorem 5 is proven.

THEOREM 6. *For* $1 \leq p, q < \infty$,

$$\left(\int_{C_0[0,1]} \|T_m f - J_{n,m}^{(d)}(f)\|_q^p w(df) \right)^{1/p} = O(n^{-m-1/2}), \tag{4.28}$$

and for $2 \leq p, q < \infty$,

$$\left(\int_{C_0[0,1]} \|T_m f - J_{n,m}^{(d)}(f)\|_q^p w(df) \right)^{1/p} \asymp n^{-m-1/2}. \tag{4.29}$$

Proof. Similar to the Proof of Theorem 1, without loss of generality we have only to prove the theorem for the case $p = q$. Let

$$
\begin{aligned}
R_n(x) &= \int_{C_0[0,1]} |T_m f(x) - J_{n,m}^{(d)}(f, x)|^2 w(df) \\
&\leq \left(\frac{1}{2}\right)^{2m+2} \int_{-\pi}^{\pi} \int_{u_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{u_2}^{u_3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{v_{2m}}^{t} \cdots \int_{-\pi}^{\pi} \int_{v_2}^{v_3} \int_{-\pi}^{\pi} \\
&\quad \times [|\cos u_1 - \cos v_0| + |\cos u_0 - \cos v_1| - |\cos u_1 - \cos v_1| \\
&\quad - |\cos u_0 - \cos v_0|] \prod_{j=0}^{m-1} \sin u_{2j+1} \sin v_{2j+1} K_{nr}(u_1, du_0) \\
&\quad \times K_{nr}(v_1, dv_0)\, du_1\, dv_1 K_{nr}(u_3, du_2) K_{nr}(v_3, dv_2) \cdots du_{2m-1} \\
&\quad \times dv_{2m-1} K_{nr}(t, du_{2m}) K_{nr}(t, dv_{2m}).
\end{aligned}
$$

By (4.21) and (4.22), there exists an absolute constant $M > 0$ such that

$$R_n(x) \leq Mn^{-2m-1} \qquad \forall x \in [0, 1].$$

Thus

$$
\begin{aligned}
\int_{C_0[0,1]} \|T_m f - J_{n,m}^{(d)}(f)\|_p^p w(df) &= \pi^{-1/2} 2^{p/2} \Gamma\left(\frac{p+1}{2}\right) \int_0^1 (R_n(x))^{p/2}\, dx \\
&= O(n^{-2m-1})^{p/2} \qquad (n \to \infty).
\end{aligned}
$$

Thus (4.28) holds; (4.29) is an immediate consequence of (4.28) and (4.14). This finishes the proof of Theorem 6.

Remark. Let $2y_k = \cos((n-k)\pi/n) + 1$, $k = 0, 1, 2, \cdots, n$, then the discrete Jackson-type operator $J_{n,m}^{(d)}$ is an algorithm based on the standard information $N_n^{**}(f) = [T_k(y_j) : 0 \leq k \leq m, 1 \leq j \leq n]$.

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